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A Generalization of the Minkowski Inequality

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1. INTRODUCTION

Let $\mathbb{R}$, $\mathbb{R}_+$, and $\mathbb{N}$ be the set of real numbers, positive numbers, and natural numbers, respectively.

If $\mathbf{x} = (x_1, ..., x_n)$, $\mathbf{y} = (y_1, ..., y_n) \in \mathbb{R}_+^n$, $n \in \mathbb{N}$, then by the classical inequality of Minkowski

$$M_{n,a}(\mathbf{x}+\mathbf{y}) \leqslant M_{n,a}(\mathbf{x}) + M_{n,a}(\mathbf{y}), \tag{1.1}$$

where $a \geqslant 1$, $\mathbf{x}+\mathbf{y} := (x_1+y_1, ..., x_n+y_n)$ and

$$M_{n,a}(\mathbf{x}) := \left(\frac{1}{n}\sum_{i=1}^{n} x_i^a\right)^{1/a} \tag{1.2}$$

is the well-known power mean of $x_1, ..., x_n$ (see Hardy *et al.* [6]).

Let $a, p \in \mathbb{R}$, $\mathbf{x} = (x_1, ..., x_n) \in \mathbb{R}_+^n$ and

$$\begin{aligned} M_{n,a}(\mathbf{x})_p &:= \left(\sum_{i=1}^{n} x_i^{a+p} \Big/ \sum_{i=1}^{n} x_i^p\right)^{1/a}, & a \neq 0, \\ &:= \exp\left(\sum_{i=1}^{n} \ln x_i \cdot x_i^p \Big/ \sum_{i=1}^{n} x_i^p\right), & a = 0. \end{aligned} \tag{1.3}$$

The inequality

$$M_{n,a}(\mathbf{x}+\mathbf{y})_p \leqslant M_{n,a}(\mathbf{x})_p + M_{n,a}(\mathbf{y})_p, \quad (\mathbf{x}, \mathbf{y} \in \mathbb{R}_+^n, n \in \mathbb{N}) \tag{1.4}$$

which may be regarded as a generalization of (1.1) was studied first by Beckenbach [1] in the case $a = 1$. He proved by the method of quasilinearization that (1.4) is valid if $a = 1$, $0 \leqslant p \leqslant 1$. Dresher [5] proved that (1.4) holds if $0 \leqslant p \leqslant 1 \leqslant a + p$. The same result was also obtained by Danskin [2] and Daróczy [3] using other known inequalities. Necessary and sufficient conditions for (1.4) and for the reversed inequality were found by

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Losonczi [7]. Losonczi [8] gave necessary and sufficient conditions for (1.4) also in the case when the range of $\mathbf{x}$, $\mathbf{y}$ is only the subset $(m, M)^n \subseteq \mathbb{R}_+^n$ $(0 \leqslant m < M < \infty)$.

In this paper we investigate the following more general inequality:

$$M_{n,a}(\mathbf{x}_1 + \cdots + \mathbf{x}_k)_p \leqslant M_{n,b_1}(\mathbf{x}_1)_{q_1} + \cdots + M_{n,b_k}(\mathbf{x}_k)_{q_k} \qquad (\mathbf{x}_1, ..., \mathbf{x}_k \in \mathbb{R}_+^n, n \in \mathbb{N}) \tag{1.5}$$

and its inverse. We give necessary and sufficient conditions concerning the parameters (a, p), (b_1, q_1),..., $(b_k, q_k) \in \mathbb{R}^2$, $k \in \mathbb{N}$ for (1.5) to hold.

2. Notations and Auxiliary Results

For $x \in \mathbb{R}$, let

$$x^+ := \max\{x, 0\}, \qquad x^- := \max\{-x, 0\} \tag{2.1}$$

and for $a, p \in \mathbb{R}$

$$\begin{aligned} j_{a,p}(t) &:= t^p(t^a - 1)/a, \qquad && a \neq 0, \quad t \in \mathbb{R}_+, \\ &:= t^p \ln t, && a = 0, \quad t \in \mathbb{R}_+. \end{aligned} \tag{2.2}$$

The following two lemmas are known:

Lemma 2.1 (Daróczy and Losonczi [4], Losonczi [8]). *Let a, b, p, $q \in \mathbb{R}$. Then the inequality*

$$j_{a,p}(t) \leqslant j_{b,q}(t) \tag{2.3}$$

is valid for $t \in \mathbb{R}$ if and only if

$$p + a^+ \leqslant q + b^+ \qquad \text{and} \qquad p - a^- \leqslant q - b^-. \tag{2.4}$$

Lemma 2.2 (Losonczi [7].) *Let $a, p \in \mathbb{R}$. The function $j_{a,p}$ is convex if and only if*

$$0 \leqslant p - a^- \leqslant 1 \leqslant p + a^+. \tag{2.5}$$

The following theorem is a simple consequence of a result of Losonczi [7, Satz 2.]; therefore, its proof is omitted.

Theorem 2.3. *Let a, b_1,..., b_k, p, q_1,..., $q_k \in \mathbb{R}$, $k \in \mathbb{N}$. The inequality*

$$M_{n,a}(\mathbf{x}_1 + \cdots + \mathbf{x}_k) \underset{(\geqslant)}{\leqslant} M_{n,b_1}(\mathbf{x}_1)_{q_1} + \cdots + M_{n,b_k}(\mathbf{x}_k)_{q_k} \tag{2.6}$$

is satisfied for every $\mathbf{x}_1,\dots,\mathbf{x}_k\in\mathbb{R}_+^n$, $n\in\mathbb{N}$ *if and only if for each* $x_1,\dots,x_k$, $y_1,\dots,y_k\in\mathbb{R}_+$

$$(y_1+\cdots+y_k)j_{a,p}(x_1+\cdots+x_k/y_1+\cdots+y_k) \underset{(\geqslant)}{\leqslant} y_1 j_{b_1,q_1}(x_1/y_1)+\cdots+y_k j_{b_k,q_k}(x_k/y_k). \tag{2.7}$$

3. The Main Results

THEOREM 3.1. *Let* $a, b_1,\dots,b_k, p, q_1,\dots,q_k\in\mathbb{R}$, $k\geqslant 2$. *The inequality*

$$M_{n,a}(\mathbf{x}_1+\cdots+\mathbf{x}_k)_p\leqslant M_{n,b_1}(\mathbf{x}_1)_{q_1}+\cdots+M_{n,b_k}(\mathbf{x}_k)_{q_k} \tag{3.1}$$

is valid for every $\mathbf{x}_1,\dots,\mathbf{x}_k\in\mathbb{R}_+^n$, $n\in\mathbb{N}$ *if and only if the inequalities*

$$\max\{p+a^+,1\}\leqslant q_i+b_i^+ \tag{3.2}$$

and

$$\max\{p-a^-,0\}\leqslant\min\{q_i-b_i^-,1\} \tag{3.3}$$

are satisfied for all $i=1,\dots,k$.

Proof. Apply Theorem 2.3 for the inequality (3.1). Substituting

$$u_i=x_i/y_i,\qquad \lambda_i=y_i/(y_1+\cdots+y_k)\qquad (i=1,\dots,k)$$

into (2.7) we get that (3.1) is valid if and only if the inequality

$$j_{a,p}(\lambda_1u_1+\cdots+\lambda_ku_k)\leqslant\lambda_1 j_{b_1,q_1}(u_1)+\cdots+\lambda_k j_{b_k,q_k}(u_k) \tag{3.4}$$

holds for each $u_1,\dots,u_k\in\mathbb{R}_+$, $\lambda_1,\dots,\lambda_k\in[0,1]$, $\sum_{i=1}^k\lambda_i=1$.

First we prove that the conditions (3.2), (3.3) are necessary. By the symmetry it is enough to show (3.2) and (3.3) in the case $i=1$.

With $\lambda_1=1$, $\lambda_2=\cdots=\lambda_k=0$ we get from (3.4) that

$$j_{a,p}(u_1)\leqslant j_{b_1,q_1}(u_1)\qquad (u_1\in\mathbb{R}_+).$$

Using Lemma 2.1 we obtain

$$p+a^+\leqslant q_1+b_1^+\qquad\text{and}\qquad p-a^-\leqslant q_1-b_1^-. \tag{3.5}$$

Put $u_2<1<u_1$, $u_3,\dots,u_k\in\mathbb{R}_+$

$$\lambda_1=(1-u_2)/(u_1-u_2),\qquad \lambda_2=1-\lambda_1,\qquad \lambda_3=\cdots=\lambda_k=0$$

into (3.4). We get

$$0 \leqslant (1-u_2)j_{b_1,q_1}(u_1) + (u_1-1)j_{b_2,q_2}(u_2)$$

that is

$$j_{b_1,q_1}(u_1)/(u_1-1) \geqslant j_{b_2,q_2}(u_2)/(u_2-1) = (j_{b_2,q_2}(u_2) - j_{b_2,q_2}(1))/(u_2-1).$$

Taking the limit $u_2 \to 1-0$ we have

$$u_1 - 1 \leqslant j_{b_1,q_1}(u_1) \qquad \text{for} \quad u_1 > 1. \tag{3.6}$$

Putting $u_1 < 1 < u_2$ into (3.4) it can be seen analogously that (3.6) is valid for $u_1 < 1$, also. Thus

$$j_{1,0}(u_1) \leqslant j_{b_1,q_1}(u_1) \qquad (u_1 \in \mathbb{R}_+). \tag{3.7}$$

By Lemma 2.1 we get

$$1 \leqslant q_1 + b_1^+ \qquad \text{and} \qquad 0 \leqslant q_1 - b_1^-. \tag{3.8}$$

Inequalities (3.8) imply that $\lim_{u_1 \to 0} j_{b_1,q_1}(u_1) =: j_{b_1,q_1}(0)$ exists and is an element of $[-1, 0]$.

Now let $\lambda \in [0, 1]$ and put $\lambda_1 = 1 - \lambda$, $\lambda_2 = \lambda$, $\lambda_3 = \cdots = \lambda_k = 0$, $u_1 = 0$, and $u_2, \ldots, u_k \in \mathbb{R}_+$ into (3.4). Then

$$j_{a,p}(\lambda u_2) \leqslant (1-\lambda)j_{b_1,q_1}(0) + \lambda j_{b_2,q_2}(u_2) \leqslant \lambda j_{b_2,q_2}(u_2). \tag{3.9}$$

We show that the inequalities

$$a + p > 1, \qquad p > 1$$

simultaneously do not hold. Suppose on the contrary that they are valid. Then $j_{a,p}(0) = 0$, $j'_{a,p}(0) = 0$ and on the other hand by (3.9)

$$j_{a,p}(\lambda u_2)/\lambda \leqslant j_{b_2,q_2}(u_2) \qquad (u_2 \in \mathbb{R}_+).$$

Taking the limit $\lambda \to 0$ we get

$$0 \leqslant j_{b_2,q_2}(u_2) \qquad (u_2 \in \mathbb{R}_+).$$

But $j_{b_2,q_2}(u_2) < 0$ if $0 < u_2 < 1$. This contradiction shows that

$$\min\{a + p, p\} = p - a^- \leqslant 1. \tag{3.10}$$

From relations (3.5), (3.8), and (3.10) the necessity of (3.2) and (3.3) follows.

Now we show that (3.2) and (3.3) are sufficient conditions. Let

$$c := \max\{p+a^+, 1\} - \max\{p-a^-, 0\}, \qquad r := \max\{p-a^-, 0\}.$$

Then by (3.2) and (3.3) $c \geqslant 0$, further

$$p+a^+ \leqslant r+c^+ = r+c, \qquad p-a^- \leqslant r-c^- = r, \tag{3.11}$$

$$r+c^+ \leqslant q_i+b_i^+, \qquad r-c^- \leqslant q_i-b_i^-, \tag{3.12}$$

and

$$0 \leqslant r-c^- \leqslant 1 \leqslant r+c^+. \tag{3.13}$$

Thus by Lemma 2.1,

$$j_{a,p}(t) \leqslant j_{c,r}(t) \leqslant j_{b_i,q_i}(t) \qquad (t \in \mathbb{R}_+) \tag{3.14}$$

and by Lemma 2.2, $j_{c,r}$ is a convex function. Hence for

$$u_1,..., u_k \in \mathbb{R}_+, \qquad \lambda_1,..., \lambda_k \in [0, 1], \qquad \sum_{i=1}^{k} \lambda_i = 1$$

we get

$$\begin{aligned} j_{a,p}(\lambda_1 u_1 + \cdots + \lambda_k u_k) &\leqslant j_{c,r}(\lambda_1 u_1 + \cdots + \lambda_k u_k) \\ &\leqslant \lambda_1 j_{c,r}(u_1) + \cdots + \lambda_k j_{c,r}(u_k) \\ &\leqslant \lambda_1 j_{b_1,q_1}(u_1) + \cdots + \lambda_k j_{b_k,q_k}(u_k) \end{aligned}$$

which completes the proof.

By Theorem 3.1, (3.14) is equivalent to the inequality

$$M_{n,a}(\mathbf{x})_p \leqslant M_{n,c}(\mathbf{x})_r \leqslant M_{n,b_i}(\mathbf{x})_{q_i} \qquad (\mathbf{x} \in \mathbb{R}_+^n, n \in \mathbb{N}) \tag{3.15}$$

and from convexity of $j_{c,r}$ it follows that

$$M_{n,c}(\mathbf{x}+\mathbf{y}) \leqslant M_{n,c}(\mathbf{x}) + M_{n,c}(\mathbf{y})_r \qquad (\mathbf{x}, \mathbf{y} \in \mathbb{R}_+^n, n \in \mathbb{N}). \tag{3.16}$$

Thus Theorem 3.1 can be stated as follows:

THEOREM 3.2. *Let* $a, b_1,..., b_k, p, q_1,..., q_k \in \mathbb{R}$, $k \geqslant 2$. *Inequality* (3.1) *holds for every* $\mathbf{x}_1,..., \mathbf{x}_k \in \mathbb{R}_+^n$, $n \in \mathbb{N}$ *if and only if there exists constants* $c, r \in \mathbb{R}$ *such that inequalities* (3.15) *and* (3.16) *are satisfied.*

THEOREM 3.3. *Let* $a, b_1,..., b_k, p, q_1,..., q_k \in \mathbb{R}$, $k \geqslant 2$. *The inequaliy*

$$M_{n,a}(\mathbf{x}_1 + \cdots + \mathbf{x}_k)_p \geqslant M_{n,b_1}(\mathbf{x}_1)_{q_1} + \cdots + M_{n,b_k}(\mathbf{x}_k)_{q_k} \tag{3.17}$$

is valid for every $\mathbf{x}_1,\ldots,\mathbf{x}_k \in \mathbb{R}_+^n$, $n \in \mathbb{N}$ *if and only if the inequalities*

$$\min\{p + a^+, 1\} \geqslant \max\{q_i + b_i^+, 0\} \tag{3.18}$$

and

$$\min\{p - a^-, 0\} \geqslant q_i - b_i^- \tag{3.19}$$

hold for all $i = 1,\ldots,k$.

Proof. By Theorem 2.3, (3.17) is valid if and only if

$$(y_1 + \cdots + y_k) j_{a,p}\left(\frac{x_1 + \cdots + x_k}{y_1 + \cdots + y_k}\right) \geqslant \sum_{i=1}^{k} y_i j_{b_i,q_i}\left(\frac{x_i}{y_i}\right) \tag{3.20}$$

for $x_1,\ldots,x_k, y_1,\ldots,y_k \in \mathbb{R}_+$.

Using the identity

$$j_{c,r}(x) = -x j_{-c,1-r}(1/x) \qquad (x \in \mathbb{R}_+, c, r \in \mathbb{R})$$

we obtain that (3.20) is equivalent to the inequality

$$(x_1 + \cdots + x_k) j_{-a,1-p}\left(\frac{y_1 + \cdots + y_k}{x_1 + \cdots + x_k}\right) \leqslant \sum_{i=1}^{k} x_i j_{-b_i,1-q_i}\left(\frac{y_i}{x_i}\right). \tag{3.21}$$

Applying Theorem 2.3, again we see that (3.21) is the necessary and sufficient condition for the inequality

$$M_{n,-a}(\mathbf{x}_1 + \cdots + \mathbf{x}_k)_{1-p} \leqslant M_{n,-b_1}(\mathbf{x}_1)_{1-q_1} + \cdots + M_{n,-b_k}(\mathbf{x}_k)_{1-q_k}$$
$$(\mathbf{x}_1,\ldots,\mathbf{x}_k \in \mathbb{R}_+^n, n \in \mathbb{N}). \tag{3.22}$$

By Theorem 3.1, the criterion of (3.22) is the system of inequalities

$$\max\{1 - p + (-a)^+, 1\} \leqslant 1 - q_i + (-b_i)^+,$$
$$\max\{1 - p - (-a)^-, 0\} \leqslant \min\{1 - q_i - (-b_i)^-, 1\} \tag{3.23}$$

($i = 1,\ldots,k$). A simple calculation shows that (3.23) is equivalent to conditions (3.18) and (3.19). Thus the proof is complete.

We remark that conditions (3.18), (3.19) are sufficient and necessary to the existence of constants $c, r \in \mathbb{R}$ for which (3.15) and (3.16) with reversed inequality signs hold.

Applying the method of Losonczi [8] we can easily obtain integral analogues of inequalities (3.1) and (3.17).

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